[4] Fung, Y. C., "On two-dimensional panel flutter," J. Aeronaut. Sci. **25**, 145–160 (1958).

[5] Voss, H. M., "On use of the reverse flow theorem in the problem of aeroelastic deflections," J. Aeronaut. Sci. 21, 569 (1954).

[6] Stearman, R., "Research on panel flutter of cylindrical shells," Air Force Office of Scientific Research, Final Scientific TR 64-0074, Midwest Research Institute (February 1964); also Graduate Aeronautical Lab., California Institute of Technology, Air Force Office of Scientific Research Rept. 3121 (July 1962).

[7] Hedgepeth, J. M., "Flutter of rectangular simply supported panels at high supersonic speeds," J. Aeronaut. Sci. **24**, 563–573 (1957).

[8] Bohon, H. L. and Dixon, S. C., "Some recent developments in flutter of flat panels," *AIAA Fifth Annual Structures and Materials Conference* (American Institute of Aeronautics and Astronautics, New York, 1964), pp. 269–281.

[9] Duncan, W. J., "Galerkin's method in mechanics and differential equations," Aeronautical Research Council R&M 1798 (1937).

[10] Miles, J. W., "Vibrations of beams on many supports," Proc. Am. Soc. Civil Engrs. **83**, EM-1, 1–4 (1956).

[11] Franklin, J. N., "On the numerical solution of characteristic equations in flutter analysis," J. Assoc. Computing Machinery **5**, 45–51 (1958).

AUGUST 1964 AIAA JOURNAL VOL. 2, NO. 8

Random Vibrations of a Myklestad Beam

Y. K. Lin*
University of Illinois, Urbana, Ill.

The stationary solution is obtained for the response of a Myklestad beam under stationary random excitations. The term response here refers to either deflection, slope, moment, or shear at different stations along the beam, and the solution is given in terms of power spectrums and cross-power spectrums. Both structural damping and viscous damping are considered. Since the transfer matrix technique is employed in the formulation, the general method developed can be extended to various types of structures whose transfer matrices are known.

Introduction

THIS paper presents a solution for the stationary random vibration of a Myklestad beam under the excitations of stationary random forces. The random forcing functions are specified in terms of their power spectrums and cross-power spectrums.

The Myklestad beam, as shown in Fig. 1, consists of piecewise uniform massless segments joined by concentrated masses. Although Fig. 1 depicts a beam of a cantilever type, it will be clear in the following analysis that other boundary conditions can be treated in an analogous manner. Such a structural model is a convenient approximation for a beam with nonuniform cross sections such as is frequently encountered in the flight vehicle structures.

The random forces are assumed to be perpendicular to the axis of the beam and concentrated at the concentrated masses. These random forces may approximate a distributed load, random in both time and space. The analysis can easily be modified for other types of excitations, for example, for random moments or for both random vertical forces and moments. However, in order to be more specific and brief, the present formulation will be for a cantilever beam and for vertical excitations.

The analysis of a Myklestad beam can best be carried out using the method of transfer matrices.[1] Therefore, a brief account will first be given on the transfer matrices applicable to the present problem. Then the power spectrums and cross-power spectrums of the stationary response will be obtained in terms of those of the exciting forces and the elements of the transfer matrices.

Received January 30, 1964; revision received May 14, 1964. This work was partially supported by Air Force Materials Laboratory, Research and Technology Division, Wright-Patterson Air Force Base, Ohio, under Contract AF 33(657)-11715.

* Associate Professor of Aeronautical and Astronautical Engineering. Member AIAA.

Transfer Matrices

In a beam problem, a transfer matrix relates the defection w, slope ϕ, moment M, and shear V at a station of the beam to those at another station. Consider a typical segment of the beam from the right of station j-1 to the left of station j, as shown in Fig. 2. It can be shown by use of elementary strength of materials techniques that the state vector (w, ϕ, M, V) on the left of station j is related to that on the right of station j-1 as follows:

$$\begin{Bmatrix} w \\ \phi \\ M \\ V \end{Bmatrix}_j^L = \begin{bmatrix} 1 & l_j & -\dfrac{l_j^2}{2(EI)_j} & -\dfrac{l_j^3}{6(EI)_j} \\ 0 & 1 & -\dfrac{l_j}{(EI)_j} & -\dfrac{l_j^2}{2(EI)_j} \\ 0 & 0 & 1 & l_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w \\ \phi \\ M \\ V \end{Bmatrix}_{j-1}^R \tag{1}$$

The square matrix in Eq. (1) is known as a field transfer matrix. In a dynamic problem, where the motion is simple harmonic motion, each element in a state vector $\{w, \phi, M, V\}$ denotes a complex amplitude. The structural damping in

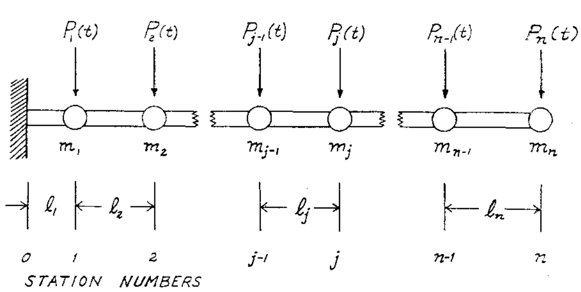


Fig. 1 A loaded Myklestad beam.

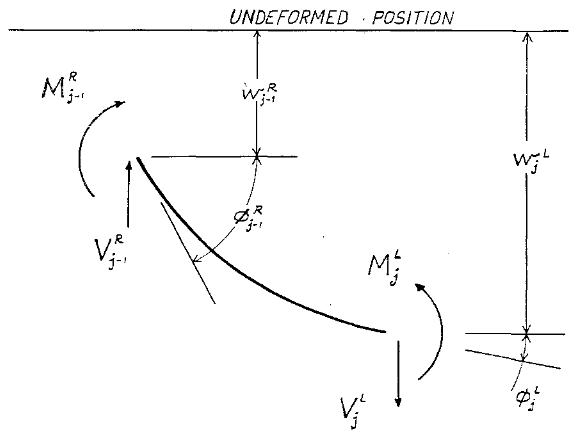


Fig. 2 Deformation of a typical segment in a Myklestad beam.

this segment can be accounted for by modifying the field transfer matrix as follows:

$$\mathbf{F}_j = \begin{bmatrix} 1 & l_j & -l_j^2/[2(EI)_j(1+i\alpha_j)] & -l_j^3/[6(EI)_j(1+i\alpha_j)] \\ 0 & 1 & -l_j/[(EI)_j(1+i\alpha_j)] & -l_j^2/[2(EI)_j(1+i\alpha_j)] \\ 0 & 0 & 1 & l_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where $i = (-1)^{1/2}$ and α_j is the structural damping factor. It is seen that matrix $\mathbf{F}$ is independent of the frequency of vibration. This is because the beam segment considered is assumed to be massless. Denoting $\mathbf{Z} = (w, \phi, M, V)$, Eq. (1) may be abbreviated

$$\mathbf{Z}_j^L = \mathbf{F}_j\, \mathbf{Z}_{j-1}^R \tag{3}$$

The relation between $\mathbf{Z}_j^R$ and $\mathbf{Z}_j^L$ may be obtained by referring to Fig. 3. If the frequency of vibration is ω, then the inertia force $m_j\ddot{w}_j = -\omega^2 m_j w_j$ and the viscous damping force $c_j\dot{w}_j = i\omega c_j w_j$. Write

$$\mathbf{Z}_j^R = \mathbf{G}_j\mathbf{Z}_j^L + \mathbf{L}_j \tag{4}$$

It is clear that

$$\mathbf{G}_j = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -m_j\omega^2 + ic_j\omega & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

and, when the excitation is just a vertical harmonic force $P_j \exp(i\omega t)$, the column

$$\mathbf{L}_j = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -P_j \end{Bmatrix} \tag{6}$$

The matrix $\mathbf{G}_j$ is generally known as a point transfer matrix.

It is interesting to note that both matrices $\mathbf{F}_j$ and $\mathbf{G}_j$ are symmetrical with respect to its cross diagonal, typical of transfer matrices in one-dimensional structures.

Power Spectrums and Cross-Power Spectrums of the Response

Depending upon the nature of a practical problem, the quantity of interest may be either a deflection, a slope, a moment, or a shear at one or several locations of the beam. Referring the desired quantity R as the response of the beam under random excitation, the statistical solution to the problem is often expressed in terms of the cross-power spectrum of the random response at two arbitrary stations. When the two stations coincide, the cross-power spectrum reduces to the power spectrum. It is well known that the autocorrelation function and the cross-correlation function are Fourier transforms of the power spectrum and the cross-power spectrum, respectively.

Let $\bar{R}_j(\omega, T)$ be the truncated Fourier transform of the response at station j:

$$\bar{R}_j(\omega, T) = \frac{1}{2\pi}\int_{-T}^{T} R_j(t)e^{-i\omega t}\, dt \tag{7}$$

The cross-power spectrum may be defined as follows:

$$\Phi_{R_jR_k}(\omega) = \lim_{T\to\infty} E\,[\pi\bar{R}_j(\omega,T)\bar{R}_k^*(\omega, T)/T] \tag{8}$$

where the asterisk denotes the complex conjugate, and the symbol $E[\;\;]$ indicates the stochastic average or mathematical expection.

Now the truncated Fourier transform of the response is related to the truncated Fourier transforms of the forcing processes as follows:

$$\bar{R}_j(\omega, T) = \sum_{k=1}^{n} H_{jk}(\omega)\bar{P}_k(\omega, T) \tag{9}$$

where $H_{jk}(\omega)$ is the complex response amplitude at station j due to a unit harmonic excitation of frequency ω at station k. This function is generally known as a frequency response function or transfer function. Substitution of Eq. (9) into Eq. (8) results in

$$\Phi_{R_jR_k} = \sum_{l=1}^{n}\sum_{s=1}^{n} H_{jl}(\omega)H_{ks}^*(\omega)\Phi_{P_lP_s}(\omega) \tag{10}$$

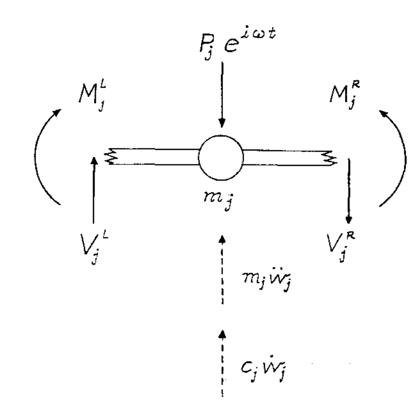
where $\Phi_{P_lP_s}$ is the cross-power spectrum of the random forces at stations l and s.

Equation (10) shows how the response statistics are related to the characteristics of the structure and the statistics of the random forces; the former are represented by the H functions and the latter by the $\Phi_{P_lP_s}$ functions.

Determination of *H* Functions

The frequency response function $H_{jl}(\omega)$ can be computed from the transfer matrices $\mathbf{F}$ and $\mathbf{G}$ defined in Eqs. (2) and (5). To illustrate, let j be the station on the right of mass j and consider the following cases: 1) $l > j$; 2) $l < j$; and 3)

Fig. 3 Force system at joint *j*; inertia and damping forces denoted by dotted lines.



$l = j$. Let ${}_l\mathbf{T}_m$ denote a 4×4 matrix obtained from the chain multiplication

$$ {}_l\mathbf{T}_m = \mathbf{G}_l\mathbf{F}_l\mathbf{G}_{l-1}\mathbf{F}_{l-1} \ldots \mathbf{G}_m\mathbf{F}_m \tag{11}$$

With only one exciting force $P_l = \exp(i\omega t)$, there exists the relation

$$\begin{Bmatrix} w \\ \phi \\ 0 \\ 0 \end{Bmatrix}_n^R = {}_n\mathbf{T}_1 \begin{Bmatrix} 0 \\ 0 \\ M \\ V \end{Bmatrix}_0^R + {}_n\mathbf{T}_{l+1} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{Bmatrix} \tag{12}$$

The boundary conditions $M_n{}^R = V_n{}^R = w_0{}^R = \phi_0{}^R = 0$ have been inserted in Eq. (12). Now

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = {}_n\begin{bmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1 \begin{Bmatrix} M \\ V \end{Bmatrix}_0^R - {}_n\begin{Bmatrix} t_{34} \\ t_{44} \end{Bmatrix}_{l+1} \tag{13}$$

can be extracted from Eq. (12), where t_{jk} represents the (j, k) element taken from the indicated T matrix [the indices at the lower corners of the matrix signify which T matrix constructed according to the chain rule of Eq. (11)]. Solving for $\{M, V\}_0{}^R$,

$$\begin{Bmatrix} M \\ V \end{Bmatrix}_0^R = {}_n\begin{bmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1^{-1} {}_n\begin{Bmatrix} t_{34} \\ t_{44} \end{Bmatrix}_{l+1} \tag{14}$$

In case 1, $l > j$, there is also the relation

$$\begin{Bmatrix} w \\ \phi \\ M \\ V \end{Bmatrix}_j^R = {}_j\mathbf{T}_1 \begin{Bmatrix} 0 \\ 0 \\ M \\ V \end{Bmatrix}_0^R = {}_j\begin{bmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \\ t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1 \begin{Bmatrix} M \\ V \end{Bmatrix}_0^R \tag{15}$$

Substituting Eq. (14) into Eq. (15)

$$\begin{Bmatrix} w \\ \phi \\ M \\ V \end{Bmatrix}_j^R = {}_j\begin{bmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \\ t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1 {}_n\begin{bmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1^{-1} {}_n\begin{Bmatrix} t_{34} \\ t_{44} \end{Bmatrix}_{l+1} \tag{16}$$

for $l > j$. This equation gives the deflection, slope, moment, and shear on the right of station j due to a unit sinusoidal excitation at station l. By definition, these are the H_{jl} functions, each of which corresponds to one type of response being considered.

In case 2, $l < j$, Eq. (15) must be replaced by

$$\begin{Bmatrix} w \\ \phi \\ M \\ V \end{Bmatrix}_j^R = {}_j\mathbf{T}_1 \begin{Bmatrix} 0 \\ 0 \\ M \\ V \end{Bmatrix}_0^R + {}_j\mathbf{T}_{l+1} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{Bmatrix} \tag{17}$$

Thus,

$$\begin{Bmatrix} w \\ \phi \\ M \\ V \end{Bmatrix}_j^R = {}_j\begin{bmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \\ t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1 {}_n\begin{bmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1^{-1} {}_n\begin{Bmatrix} t_{34} \\ t_{44} \end{Bmatrix}_{l+1} - {}_j\begin{Bmatrix} t_{14} \\ t_{24} \\ t_{34} \\ t_{44} \end{Bmatrix}_{l+1} \quad \text{for } l < j \tag{18}$$

Finally, it can easily be seen that the validity of Eq. (18) can be extended to case 3, $l = j$, if the matrix ${}_j\mathbf{T}_{j+1}$ is replaced by an identity matrix.† Once the frequency response functions are known, the cross-power spectrum of the response at any two stations is readily obtained from Eq. (10).

† This is analogous to $0! = 1$.

Power Spectrums of the Response at Critical Locations

Although the method outlined in the foregoing paragraphs permits the evaluation of cross-power spectrum for any of the four types of response at any two stations, the general procedure may be quite tedious when the number of segments in a Myklestad beam is large. In many practical problems, however, it may be only necessary to determine the response at certain critical locations. For example, for a cantilever beam such as the one being discussed, it may be sufficient to know the statistics of the random moment and shear at the clamped support and/or of the random deflection and slope at the free end. One of the random quantities just mentioned may be the controlling factor in a design problem. As it often happens, the power spectrums of the response at such critical locations can be more simply obtained without following the general procedure.

Let it be required to compute the power spectrums of the moment and the shear at the clamped support $\{M, V\}_0{}^R$, under the excitations of random forces $P_1(t)$, $P_2(t)$, . . . , $P_n(t)$. Since excitations are present at all stations 1, 2, . . . n, the following relation may be written for the truncated Fourier transforms of the random functions in question:

$$\begin{Bmatrix} \bar{w} \\ \bar{\phi} \\ 0 \\ 0 \end{Bmatrix}_n^R = {}_n\mathbf{T}_1 \begin{Bmatrix} 0 \\ 0 \\ \bar{M} \\ \bar{V} \end{Bmatrix}_0^R + \sum_{l=1}^{n} {}_n\mathbf{T}_{l+1} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\bar{P}_l \end{Bmatrix} \tag{19}$$

where a bar denotes a truncated Fourier transformation as defined in Eq. (7), and it is understood that ${}_n\mathbf{T}_{n+1} = \mathbf{I}$. Equation (19) may be separated into two parts as follows:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = {}_n\begin{bmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1 \begin{Bmatrix} \bar{M} \\ \bar{V} \end{Bmatrix}_0^R - \sum_{l=1}^{n} {}_n\begin{Bmatrix} t_{34} \\ t_{44} \end{Bmatrix}_{l+1} \bar{P}_l \tag{20}$$

and

$$\begin{Bmatrix} \bar{w} \\ \bar{\phi} \end{Bmatrix}_n^R = {}_n\begin{bmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \end{bmatrix}_1 \begin{Bmatrix} \bar{M} \\ \bar{V} \end{Bmatrix}_0^R - \sum_{l=1}^{n} {}_n\begin{Bmatrix} t_{14} \\ t_{24} \end{Bmatrix}_{l+1} \bar{P}_l \tag{21}$$

Let $\bar{\mathbf{\Pi}}$ denote the column $\{\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n\}$ and let $\mathbf{S}$ be a $2 \times n$ matrix formed by the columns

$${}_n\begin{Bmatrix} t_{34} \\ t_{44} \end{Bmatrix}_{l+1}$$

i.e.,

$$\mathbf{S} = \left[{}_n\begin{Bmatrix} t_{34} \\ t_{44} \end{Bmatrix}_2, {}_n\begin{Bmatrix} t_{34} \\ t_{44} \end{Bmatrix}_3, \ldots, {}_n\begin{Bmatrix} t_{34} \\ t_{44} \end{Bmatrix}_n, \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \right] \tag{22}$$

Then Eq. (20) can be rewritten as follows:

$$\begin{Bmatrix} \bar{M} \\ \bar{V} \end{Bmatrix}_0^R = {}_n\begin{bmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1^{-1} \mathbf{S}\,\bar{\mathbf{\Pi}} \tag{23}$$

The matrix of cross-power spectrums of $\{M, V\}_0{}^R$ follows from the application of Eq. (8):

$$\begin{bmatrix} \Phi_{MM} & \Phi_{MV} \\ \Phi_{VM} & \Phi_{VV} \end{bmatrix}_0^R = {}_n\begin{bmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1^{-1} \mathbf{S}\mathbf{\Phi}_\pi\mathbf{S}^{*\prime} \, {}_n\begin{bmatrix} t_{33}^* & t_{34}^* \\ t_{43}^* & t_{44}^* \end{bmatrix}_1^{-1\prime} \tag{24}$$

where a prime denotes a matrix transposition, and $\mathbf{\Phi}_\pi$ is the matrix of cross-power spectrums of the random forces, i.e.,

$$\mathbf{\Phi}_\pi = \begin{bmatrix} \Phi_{P_1P_1} & \Phi_{P_1P_2} \ldots & \Phi_{P_1P_n} \\ \Phi_{P_2P_1} & \Phi_{P_2P_2} \ldots & \Phi_{P_2P_n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \Phi_{P_nP_1} & \Phi_{P_nP_2} \ldots & \Phi_{P_nP_n} \end{bmatrix} \tag{25}$$

The diagonal elements on the left-hand side of Eq. (24) are the required power-spectrums of M and V at the clamped support, and the nondiagonal elements are their cross-power spectrums‡ which, however, have no physical significance in the present problem.

If the power spectrums of the deflection and slope at the free end are also required, Eq. (23) can be substituted into Eq. (21) to give

$$\left\{ \begin{matrix} \bar{w} \\ \bar{\phi} \end{matrix} \right\}_n^R = {}_n\begin{bmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \end{bmatrix}_1 {}_n\begin{bmatrix} t_{33} & t_{34} \\ t_{43} & t_{14} \end{bmatrix}_1^{-1} \mathbf{S}\,\overline{\mathbf{\Pi}} - \mathbf{V}\,\overline{\mathbf{\Pi}} \qquad (26)$$

where

$$\mathbf{V} = \left[{}_n\left\{ \begin{matrix} t_{14} \\ t_{24} \end{matrix} \right\}_2, {}_n\left\{ \begin{matrix} t_{14} \\ t_{24} \end{matrix} \right\}_3, \ldots, {}_n\left\{ \begin{matrix} t_{14} \\ t_{24} \end{matrix} \right\}_n, \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} \right] \qquad (27)$$

is a $2 \times n$ matrix formed by the columns ${}_n\{t_{14}, t_{24}\}_{i+1}$. Eq. (26) leads to

$$\begin{bmatrix} \Phi_{ww} & \Phi_{w\phi} \\ \Phi_{\phi w} & \Phi_{\phi\phi} \end{bmatrix}_n^R = {}_n\begin{bmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \end{bmatrix}_1 {}_n\begin{bmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{bmatrix}_1^{-1}$$
$$(\mathbf{S} - \mathbf{V})\mathbf{\Phi}_{\Pi}(\mathbf{S}^* - \mathbf{V}^*)$$
$${}_n\begin{bmatrix} t_{33}^* & t_{34}^* \\ t_{43}^* & t_{44}^* \end{bmatrix}_1^{-1\prime} {}_n\begin{bmatrix} t_{13}^* & t_{14}^* \\ t_{23}^* & t_{24}^* \end{bmatrix}_1^{\prime} \qquad (28)$$

‡ Note that $\Phi_{MV} = \Phi_{VM}^*$.

Concluding Remarks

The Myklestad representation of a nonuniform beam has been used widely in the analysis of aircraft structures, particularly for slender wings. The general method described in this paper can be adapted to the study of aircraft response to random gusts when the wing flexibility is a major consideration. The transfer matrix technique employed herein not only results in a neat presentation but also allows for further extension to other structural configurations for which the transfer matrices have been extensively explored.[1, 2] Recently, this technique was used to determine the basic dynamic characteristics of skin-stiffener panels,[3] which are typical in modern fuselage construction, and for which the fatigue failure due to jet noise excitation is still an annoying problem. Thus the introduction of transfer matrix formulation in random vibration analyses may also point toward a new direction for better prediction of panel response to jet noise.

References

[1] Pestel, E. G. and Leckie, F. A., *Matrix Methods in Elastomechanics* (McGraw Hill Book Co., Inc., New York, 1963).

[2] Sneddon, I. N. and Hill, R. (eds.), *Progress in Solid Mechanics* (North-Holland Publishing Co., Amsterdam, 1960), Chap. 2 pp. 61–81.

[3] Lin, Y. K., "Dynamic characteristics of continuous skin-stringer panels," *Proceedings of the Second International Conference on Acoustical Fatigue* (to be published).